# Vectors



# VECTOR PRODUCT

#### Graham S McDonald

A Tutorial Module for learning about the vector product of two vectors

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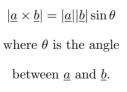
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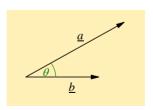
# 1. Theory

The purpose of this tutorial is to practice working out the vector product of two vectors.

It is called the 'vector product' because the result is a 'vector', i.e. a quantity with <u>both</u> (i) **magnitude**, and (ii) **direction**.

# (i) The **MAGNITUDE** of the **vector product** of $\underline{a}$ and $\underline{b}$ is









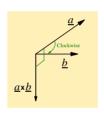






(ii) The **DIRECTION** of the **vector product** of  $\underline{a}$  and  $\underline{b}$  is **perpendicular** to both  $\underline{a}$  and  $\underline{b}$ ,

such that if we look along  $\underline{a} \times \underline{b}$  then  $\underline{a}$  rotates towards  $\underline{b}$  in a clockwise manner.



It can be shown that the above definitions of magnitude and direction of a vector product allow us to calculate the x, y and z **components** of  $\underline{a} \times \underline{b}$  from the individual components of the vectors  $\underline{a}$  and  $\underline{b}$ 











The **components of the vector**  $\underline{a} \times \underline{b}$  are given by the 'determinant' of a matrix with 3 rows.

The components of  $\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$  and  $\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$  appear in the 2nd and 3rd rows.

This 3-row determinant is evaluated by expansion into 2-row determinants, which are themselves then expanded. Matrix theory is itself very useful, and the scheme is shown below

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} 
= \underline{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \underline{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \underline{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} 
= \underline{i} (a_y b_z - a_z b_y) - \underline{j} (a_x b_z - a_z b_x) + \underline{k} (a_x b_y - a_y b_x)$$









Section 2: Exercises

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## 2. Exercises

Click on Exercise links for full worked solutions (there are 14 exercises in total).

EXERCISE 1. Calculate  $|\underline{a} \times \underline{b}|$  when  $|\underline{a}| = 2$ ,  $|\underline{b}| = 4$  and the angle between  $\underline{a}$  and  $\underline{b}$  is  $\theta = 45^{\circ}$ 

EXERCISE 2. Calculate  $|\underline{a} \times \underline{b}|$  when  $|\underline{a}| = 3$ ,  $|\underline{b}| = 5$  and the angle between  $\underline{a}$  and  $\underline{b}$  is  $\theta = 60^{\circ}$ 

EXERCISE 3. Calculate  $|\underline{a} \times \underline{b}|$  when  $|\underline{a}| = 1$ ,  $|\underline{b}| = 3$  and the angle between  $\underline{a}$  and  $\underline{b}$  is  $\theta = 30^{\circ}$ 

EXERCISE 4. Calculate  $|\underline{a} \times \underline{b}|$  when  $|\underline{a}| = 2$ ,  $|\underline{b}| = 5$  and the angle between  $\underline{a}$  and  $\underline{b}$  is  $\theta = 35^{\circ}$ 

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EXERCISE 5. Calculate the magnitude of the torque  $\underline{\tau} = \underline{s} \times \underline{F}$  when  $|\underline{s}| = 2 \,\mathrm{m}, \ |\underline{F}| = 4 \,\mathrm{N}$  and  $\theta = 30^\circ$  ( $\theta$  is the angle between the position vector  $\underline{s}$  and the force  $\underline{F}$ )

EXERCISE 6. Calculate the magnitude of the velocity  $\underline{v} = \underline{\omega} \times \underline{s}$  when  $|\underline{\omega}| = 3 \, \mathrm{s}^{-1}$ ,  $|\underline{s}| = 2 \, \mathrm{m}$  and  $\theta = 45^{\circ}$  ( $\theta$  is the angle between the angular velocity  $\underline{\omega}$  and the position vector  $\underline{s}$ )

EXERCISE 7. If  $\underline{a} = 4\underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{b} = 2\underline{i} - 6\underline{i} - 3\underline{k}$  then calculate a vector that is perpendicular to both  $\underline{a}$  and  $\underline{b}$ 

EXERCISE 8. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$  and  $\underline{b} = 3\underline{i} - 6\underline{j} + 2\underline{k}$ 

Toc









EXERCISE 9. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = 3\underline{i} + 4\underline{j} - 3\underline{k}$  and  $\underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$ 

EXERCISE 10. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{b} = 3\underline{i} + 3\underline{j} + \underline{k}$ 

EXERCISE 11. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = 2\underline{i} + 4\underline{j} + 2\underline{k}$  and  $\underline{b} = \underline{i} + 5\underline{j} - 2\underline{k}$ 

EXERCISE 12. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = 3\underline{i} - 4\underline{j} + \underline{k}$  and  $\underline{b} = 2\underline{i} + 5\underline{j} - \underline{k}$ 

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EXERCISE 13. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$  and  $\underline{b} = 2\underline{i} + \underline{j} + 4\underline{k}$ 

EXERCISE 14. Calculate the vector  $\underline{a} \times \underline{b}$  when  $\underline{a} = 2\underline{i} + 3\underline{k}$  and  $\underline{b} = \underline{i} + 2\underline{j} + 4\underline{k}$ 

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# 3. Answers

- 1.  $4\sqrt{2}$ ,
- 2.  $\frac{15}{2}\sqrt{3}$ ,
- $3. \ \frac{3}{2},$
- 4. 5.766,
- 5. 4 J,
- 6.  $3\sqrt{2} \text{ m s}^{-1}$ ,
- 7.  $-12\underline{i} + 10j 28\underline{k}$ ,
- $8. -4\underline{i} 7\underline{j} 15\underline{k},$
- 9.  $17\underline{i} 9\underline{j} + 5\underline{k}$ ,
- $10. \ 5\underline{i} 4\underline{j} 4\underline{k},$
- $11. -18\underline{i} + 6\underline{j} + 6\underline{k},$
- 12.  $-\underline{i} + 5\underline{j} + 23\underline{k}$ ,
- $13. -13\underline{i} 6\underline{j} + 8\underline{k},$
- $14. -6\underline{i} 5j + 4\underline{k}.$









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# 4. Tips on using solutions

• When looking at the THEORY, ANSWERS, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

• Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

• Try to make less use of the full solutions as you work your way through the Tutorial









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# 5. Alternative notation

- ullet Here, we use symbols like  $\underline{a}$  to denote a vector. In some texts, symbols for vectors are **in bold** (eg **a** instead of  $\underline{a}$ )
- In this Tutorial, vectors are given in terms of the unit Cartesian vectors  $\underline{i}$ ,  $\underline{j}$  and  $\underline{k}$ . A common alternative notation involves quoting the **Cartesian components** within brackets. For example, the vector  $\underline{a} = 2\underline{i} + j + 5\underline{k}$  can be written as  $\underline{a} = (2, 1, 5)$
- The scalar product  $\underline{a} \cdot \underline{b}$  is also called a 'dot product' (reflecting the symbol used to denote this type of multiplication). Likewise, the vector product  $\underline{a} \times \underline{b}$  is also called a '**cross product**'
- An alternative notation for the vector product is  $\underline{a} \wedge \underline{b}$









# Full worked solutions

## Exercise 1.

Here, we have  $|\underline{a}| = 2$ ,  $|\underline{b}| = 4$  and  $\theta = 45^{\circ}$ .

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin \theta$$

$$= (2)(4)\sin 45^{\circ}$$

$$= (2)(4)\frac{1}{\sqrt{2}}$$

$$= \frac{8}{\sqrt{2}}$$

$$= \frac{8\sqrt{2}}{\sqrt{2}}$$

$$= \frac{8\sqrt{2}}{2}$$

$$= 4\sqrt{2}.$$

Return to Exercise 1

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## Exercise 2.

Here, we have  $|\underline{a}| = 3$ ,  $|\underline{b}| = 5$  and  $\theta = 60^{\circ}$ .

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin \theta$$
$$= (3)(5)\sin 60^{\circ}$$
$$= (3)(5)\frac{\sqrt{3}}{2}$$
$$= \frac{15}{2}\sqrt{3}.$$

Return to Exercise 2









## Exercise 3.

Here, we have  $|\underline{a}| = 1$ ,  $|\underline{b}| = 3$  and  $\theta = 30^{\circ}$ .

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin \theta$$

$$= (1)(3)\sin 30^{\circ}$$

$$= (1)(3)\frac{1}{2}$$

$$= \frac{3}{2}.$$

Return to Exercise 3









### Exercise 4.

Here, we have  $|\underline{a}| = 2$ ,  $|\underline{b}| = 5$  and  $\theta = 35^{\circ}$ .

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin \theta$$

$$= (2)(5)\sin 35^{\circ}$$

$$\simeq (2)(5)(0.5736)$$

$$= 5.736.$$

Return to Exercise 4









## Exercise 5.

Here, we have  $|\underline{s}|=2\,\mathrm{m},\,|\underline{F}|=4\,\mathrm{N}$  and  $\theta=30^\circ.$ 

$$|\tau| = |\underline{s} \times \underline{F}|$$

$$= |\underline{s}||\underline{F}|\sin \theta$$

$$= (2 \text{ m})(4 \text{ N})\sin 30^{\circ}$$

$$= (2 \text{ m})(4 \text{ N})\frac{1}{2}$$

$$= (8)(\text{N m})\frac{1}{2}$$

$$= (8)(\text{J})\frac{1}{2}$$

$$= 4 \text{ J}.$$

Return to Exercise 5









#### Exercise 6.

Here, we have  $|\underline{\omega}| = 3 \text{ s}^{-1}$ ,  $|\underline{s}| = 2 \text{ m}$  and  $\theta = 45^{\circ}$ .

$$\begin{aligned} |\tau| &= |\underline{\omega} \times \underline{s}| \\ &= |\underline{\omega}||\underline{s}|\sin \theta \\ &= (3 \, \mathrm{s}^{-1})(2 \, \mathrm{m})\sin 45^{\circ} \\ &= (3 \, \mathrm{s}^{-1})(2 \, \mathrm{m})\frac{1}{\sqrt{2}} \\ &= (6)(\mathrm{m} \, \mathrm{s}^{-1})\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \\ &= (6)(\mathrm{m} \, \mathrm{s}^{-1})\frac{\sqrt{2}}{2} \\ &= 3\sqrt{2} \, \mathrm{m} \, \mathrm{s}^{-1}. \end{aligned}$$

Return to Exercise 6









#### Exercise 7.

 $\underline{a} \times \underline{b}$  is a vector that is perpendicular to both  $\underline{a}$  and  $\underline{b}$ . Here, we have  $\underline{a} = 4\underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{b} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ .

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & \overline{2} & -1 \\ 2 & -6 & -3 \end{vmatrix} 
= \underline{i} \begin{vmatrix} 2 & -1 \\ -6 & -3 \end{vmatrix} - \underline{j} \begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underline{k} \begin{vmatrix} 4 & 2 \\ 2 & -6 \end{vmatrix} 
= \underline{i}[(2)(-3) - (-1)(-6)] - \underline{j}[(4)(-3) - (-1)(2)] + \underline{k}[(4)(-6) - (2)(2)]$$

$$\underline{a} \times \underline{b} = \underline{i}[-6-6] - \underline{j}[-12+2] + \underline{k}[-24-4]$$
$$= -12\underline{i} + 10\underline{j} - 28\underline{k}.$$

Return to Exercise 7

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#### Exercise 8.

Here, we have  $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$  and  $\underline{b} = 3\underline{i} - 6\underline{j} + 2\underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 3 & -6 & 2 \end{vmatrix} 
= \underline{i} \begin{vmatrix} 1 & -1 \\ -6 & 2 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & 1 \\ 3 & -6 \end{vmatrix} 
= \underline{i}[(1)(2) - (-1)(-6)] - \underline{j}[(2)(2) - (-1)(3)] + \underline{k}[(2)(-6) - (1)(3)] 
\underline{a} \times \underline{b} = \underline{i}[2 - 6] - \underline{j}[4 + 3] + \underline{k}[-12 - 3] 
= -4\underline{i} - 7\underline{j} - 15\underline{k}.$$

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#### Exercise 9.

Here, we have  $\underline{a} = 3\underline{i} + 4\underline{j} - 3\underline{k}$  and  $\underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & \overline{4} & -3 \\ 1 & 3 & 2 \end{vmatrix} 
= \underline{i} \begin{vmatrix} 4 & -3 \\ 3 & 2 \end{vmatrix} - \underline{j} \begin{vmatrix} 3 & -3 \\ 1 & 2 \end{vmatrix} + \underline{k} \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} 
= \underline{i}[(4)(2) - (-3)(3)] - \underline{j}[(3)(2) - (-3)(1)] + \underline{k}[(3)(3) - (4)(1)] 
\underline{a} \times \underline{b} = \underline{i}[8 + 9] - \underline{j}[6 + 3] + \underline{k}[9 - 4] 
= 17\underline{i} - 9\underline{j} + 5\underline{k}.$$

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#### Exercise 10.

Here, we have  $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$  and  $\underline{b} = 3\underline{i} + 3\underline{j} + \underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \underline{i} \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}$$

$$= \underline{i}[(2)(1) - (-1)(3)] - \underline{j}[(1)(1) - (-1)(3)] + \underline{k}[(1)(3) - (2)(3)]$$
i.e.  $\underline{a} \times \underline{b} = \underline{i}[2+3] - \underline{j}[1+3] + \underline{k}[3-6]$ 

$$= 5\underline{i} - 4\underline{j} - 3\underline{k}.$$

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## Exercise 11.

Here, we have  $\underline{a} = 2\underline{i} + 4\underline{j} + 2\underline{k}$  and  $\underline{b} = \underline{i} + 5\underline{j} - 2\underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & 2 \\ 1 & 5 & -2 \end{vmatrix} 
= \underline{i} \begin{vmatrix} 4 & 2 \\ 5 & -2 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} 
= \underline{i}[(4)(-2)-(2)(5)] - \underline{j}[(2)(-2)-(2)(1)] + \underline{k}[(2)(5)-(4)(1)] 
\underline{a} \times \underline{b} = \underline{i}[-8 - 10] - \underline{j}[-4 - 2] + \underline{k}[10 - 4] 
= -18\underline{i} + 6\underline{j} + 6\underline{k}.$$

Return to Exercise 11

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### Exercise 12.

Here, we have  $\underline{a} = 3\underline{i} - 4\underline{j} + \underline{k}$  and  $\underline{b} = 2\underline{i} + 5\underline{j} - \underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 1 \\ 2 & 5 & -1 \end{vmatrix} 
= \underline{i} \begin{vmatrix} -4 & 1 \\ 5 & -1 \end{vmatrix} - \underline{j} \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + \underline{k} \begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix} 
= \underline{i}[(-4)(-1)-(1)(5)] - \underline{j}[(3)(-1)-(1)(2)] + \underline{k}[(3)(5)-(-4)(2)] 
\underline{a} \times \underline{b} = \underline{i}[4-5] - \underline{j}[-3-2] + \underline{k}[15+8] 
= -\underline{i} + 5\underline{j} + 23\underline{k}.$$

Return to Exercise 12

Toc









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#### Exercise 13.

Here, we have  $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$  and  $\underline{b} = 2\underline{i} + \underline{j} + 4\underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 1 \\ 2 & 1 & 4 \end{vmatrix} 
= \underline{i} \begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} 
= \underline{i}[(-3)(4) - (1)(1)] - \underline{j}[(2)(4) - (1)(2)] + \underline{k}[(2)(1) - (-3)(2)] 
\underline{a} \times \underline{b} = \underline{i}[-12 - 1] - \underline{j}[8 - 2] + \underline{k}[2 + 6] 
= -13\underline{i} - 6\underline{j} + 8\underline{k}.$$

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#### Exercise 14.

Here, we have  $\underline{a} = 2\underline{i} + 3\underline{k} = 2\underline{i} + 0\underline{j} + 3\underline{k}$  and  $\underline{b} = \underline{i} + 2\underline{j} + 4\underline{k}$ 

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 3 \\ 1 & 2 & 4 \end{vmatrix} 
= \underline{i} \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} 
= \underline{i}[(0)(4) - (3)(2)] - \underline{j}[(2)(4) - (3)(1)] + \underline{k}[(2)(2) - (0)(1)] 
\underline{a} \times \underline{b} = \underline{i}[0 - 6] - \underline{j}[8 - 3] + \underline{k}[4 - 0] 
= -6\underline{i} - 5\underline{j} + 4\underline{k}.$$

Return to Exercise 14

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